Manifold Relativity:
Sub-Additivity from Spectral Truncation

Preprint v13.0 — O31 Computation, O1 Constraint

Clarification, and Space–Time Complementarity

Continuing the Manifold Relativity programme
(v1–v11 appeared under the historical title "Entropy Waves, Coordinate Systems,

and the Self-Referential Universe")

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Abstract

Version 12 of the Manifold Relativity programme consolidated the framework's terminology, proved three structural propositions about spectral accessibility, and stated candidate observables with falsification criteria. It identified two frontier targets: O31 (emergence of the κ-addition composition law from spectral truncation) and O1 (derivation of the off-diagonal spatial metric component $g_{IE}(I, E)$).

This edition advances O31 computationally and clarifies O1 structurally.

First, a **convention lock** fixes the spectral filter definition and its direction once, preventing sign or cutoff ambiguity in downstream calculations.

Second, we perform an explicit computation of O31 across multiple system sizes. Composite toy Dirac operators from 4×4 to 16×16 dimensions are spectrally truncated, and the resulting entropy composition is compared against κ-addition.

Result: Spectral truncation produces sub-additive entropy composition in every bipartite test case, at every system size tested (up to 16×16). The effective κ parameter increases monotonically with the fraction of spectrum retained: more accessible spectrum means weaker apparent correlation. The full-spectrum limit recovers additive composition ($\kappa \to \infty$). However, the specific κ-addition functional form (defect proportional to $S_A \cdot S_B$) is not uniquely selected by the computation: alternative sub-additive forms fit the data comparably well. The qualitative mechanism of the v7 bridge conjecture is validated; its exact quantitative form remains open.

Third, we derive an **exact analytical formula** for the mutual information created by spectral truncation of product thermal states. The mutual information is a Jensen gap measuring the non-uniformity of spectral accessibility across subsystems. It vanishes if and only if the truncation mask is a product set. This identifies the precise mechanism: the non-product geometry of the truncation mask is the sole source of apparent correlation.

Fourth, we clarify the structure of the O1 inverse problem. A critical finding is that the vacuum Einstein equations are trivially satisfied for *any* two-dimensional metric, making the naive "backward EFE recovery" constraint vacuous when applied to the (I, E) sector alone. The actual constraints on g_{IE} are identified: the three-dimensional metric construction via the replacement product (v3), the determinant condition for spatial dimensionality, the Ryu–Takayanagi geodesic structure, and the rotation map convergence.

Fifth, we prove a Space–Time Complementarity Corollary from the v12 structural propositions: spatial spectral richness and temporal processing rate are inversely related across temperature, with explicit consequences for cycle-endpoint behaviour.

Contents

Convention Lock

All calculations in this edition use the following fixed convention, inherited from v12 Definition 2.1.

Definition 1.1 (Spectral Filter Convention — Locked)**.** The accessible spectral sector at temperature T is:

$$U_T := \{\lambda \in \mathrm{spec}(D_{\mathcal{W}}) : |\lambda| \geq k_B T/\hbar\}. \tag{1}$$

The spectral filter Π_T projects onto U_T. The convention is:

- Higher $T \implies$ higher threshold $\implies$ fewer eigenvalues kept $\implies$ smaller accessible sector.
- Lower $T \implies$ lower threshold $\implies$ more eigenvalues kept $\implies$ larger accessible sector.
- $T \to 0$: threshold $\to 0$, full spectrum accessible.
- $T \to T_P$: threshold $\to E_P/\hbar$, maximal truncation.

This convention is load-bearing for the v12 monotone filtration (Proposition 3.1), the P20 measurement-floor prediction, the P23 chart-mismatch residual, and the RG-flow conjecture (O36).

The O31 Computation: Sub-Additivity from Spectral Truncation

Setup

Open Problem O31 (v10) asks whether the H-function composition law of thermodynamic relativity [2] emerges from spectral truncation of the candidate Dirac operator. We test this in the simplest non-trivial setting.

Subsystem operators. Following v6.1, each subsystem is a two-site graph with Dirac operator

$$D_A = D_B = \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}, \tag{2}$$

where $a > 0$ is the coupling strength (identified with $c_S = k_B T/\hbar$ in v6.1). The spectrum is $\{-a, +a\}$.

Composite operator. The composite system $A \otimes B$ has Hilbert space $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2 = \mathbb{C}^4$ and Dirac operator

$$D_{AB} = D_A \otimes \mathbf{1}_B + \mathbf{1}_A \otimes D_B. \tag{3}$$

The spectrum of D_{AB} is $\{\lambda_i + \mu_j\}$ where $\lambda_i \in \{-a, +a\}$ and $\mu_j \in \{-b, +b\}$.

For the symmetric case $a = b$: $\mathrm{spec}(D_{AB}) = \{-2a, 0, 0, +2a\}$.

For the asymmetric case $a \neq b$: $\mathrm{spec}(D_{AB}) = \{-(a+b), -(a-b), +(a-b), +(a+b)\}$.

Thermal state. The thermal density matrix at inverse temperature β is

$$\rho = \frac{e^{-\beta D_{AB}}}{\mathrm{Tr}(e^{-\beta D_{AB}})}. \tag{4}$$

For a product Hamiltonian $D_{AB} = D_A \otimes \mathbf{1} + \mathbf{1} \otimes D_B$, the thermal state factors: $\rho = \rho_A \otimes \rho_B$. The von Neumann entropy is therefore *additive*: $S(\rho) = S(\rho_A) + S(\rho_B)$.

Spectral Truncation

Definition 2.1 (Spectral Truncation of a Thermal State)**.** Given a threshold $\theta > 0$, the spectral truncation Π_T retains eigenstates of D_{AB} with $|\lambda| \geq \theta$ and removes those with $|\lambda| < \theta$. The truncated state is the renormalised projection:

$$\rho_\theta := \frac{\Pi_T \, \rho \, \Pi_T}{\mathrm{Tr}(\Pi_T \, \rho \, \Pi_T)}. \tag{5}$$

For the symmetric case with $\theta > 0$ (removing the two zero-eigenvalue states), the truncated subspace is spanned by $\{|{++}\rangle, |{--}\rangle\}$ — a maximally correlated subspace. Even though the full state ρ is a product state with no correlations, the truncated state ρ_θ lives in a correlated subspace.

The truncation creates apparent correlation by discarding the uncorrelated sector.

The Entropy Defect

For the full (untruncated) state:

$$S(\rho) = S(\rho_A) + S(\rho_B) \qquad \text{(additive, zero defect)}. \tag{6}$$

For the truncated state, define the marginals $\rho_\theta^A := \mathrm{Tr}_B(\rho_\theta)$ and $\rho_\theta^B := \mathrm{Tr}_A(\rho_\theta)$. The entropy defect is:

$$\Delta S := S(\rho_\theta) - S(\rho_\theta^A) - S(\rho_\theta^B) < 0. \tag{7}$$

Result: Sub-Additivity Confirmed; Exact κ-Form Remains Open

Computation 2.2 (O31 Toy Result). For every parameter combination tested — symmetric and asymmetric couplings, multiple inverse temperatures β, multiple truncation thresholds θ, and system dimensions from 4 to 16 — spectral truncation of a product thermal state produces **sub-additive** entropy composition:

$$S(\rho_\theta) < S(\rho_\theta^A) + S(\rho_\theta^B) \qquad \text{(negative entropy defect)}. \tag{8}$$

The sub-additivity vanishes in the full-spectrum limit ($\theta \to 0$, no truncation), recovering standard additive composition.

The defect can be expressed in κ-addition form:

$$S(\rho_\theta) = S(\rho_\theta^A) + S(\rho_\theta^B) - \frac{1}{\kappa}\, S(\rho_\theta^A)\, S(\rho_\theta^B), \tag{9}$$

where $\kappa > 0$ is defined by the defect. However, this expression is formally satisfiable for *any* bipartite entropy split with nonzero marginal entropies (one free parameter, one equation). The non-trivial content is not the formal match of the formula but the following three properties:

1. **Sub-additivity:** The defect is always negative (entropy is sub-additive) for direct spectral truncation of product states. This corresponds to $\kappa > 0$.
2. **Monotonic scaling:** The effective κ increases monotonically with the fraction of spectrum retained. More accessible spectrum means weaker apparent correlation.
3. **Additivity recovery:** In the full-spectrum limit, $\kappa \to \infty$ and additive composition is recovered exactly.

Remark 2.3 (The κ-addition form is not uniquely selected). A cross-check was performed: the alternative model $\Delta S = -\alpha(S_A + S_B)$ was fitted alongside $\Delta S = -S_A S_B/\kappa$. At fixed truncation across varying β, the coefficient of variation of α (0.36) is lower than that of κ (0.55). Neither parameter is constant across β, indicating that the true functional form of the defect is more complex than either simple model. The specific κ-addition form is **consistent** with the data but is **not uniquely determined** by the computation. The emergence of the exact Livadiotis–McComas composition law from spectral truncation remains an open question.

Scaling Behaviour

The results in Table 1 reveal a robust scaling pattern:

1. **System-size robustness:** Sub-additive entropy defect appears at every dimension tested (4×4 through 16×16). It is not a small-system artifact.
2. **Monotonic κ–truncation relationship:** The empirical κ increases monotonically with the fraction of spectrum retained. For the 16-dimensional system: $n/\dim =$

Table 1: O31 computation results across system sizes. dim: Hilbert space dimension. n: eigenvalues retained. κ: empirical κ-addition parameter. All listed bipartite cases exhibit negative entropy defect; empirical κ-parameterisation shown for comparison.

System	dim	$n/$dim	S_{AB}	$S_A + S_B$	ΔS	κ
2-site ⊗ 2-site (symmetric)						
$a{=}b{=}1, \beta{=}0.5$	4	2/4	0.365	0.731	−0.365	0.37
$a{=}b{=}1, \beta{=}1.0$	4	2/4	0.090	0.180	−0.090	0.09
2-site ⊗ 2-site (asymmetric)						
$a{=}1, b{=}0.5$	4	3/4	0.714	0.758	−0.045	2.07
$a{=}1, b{=}0.3$	4	3/4	0.574	0.692	−0.118	0.90
3-site chain ⊗ 3-site chain						
	9	6/9	1.427	1.661	−0.234	2.95
	9	4/9	0.975	1.436	−0.461	1.12
	9	2/9	0.215	0.431	−0.215	0.22
4-site chain ⊗ 4-site chain						
	16	12/16	2.105	2.275	−0.171	7.58
	16	8/16	1.643	1.961	−0.317	3.03
	16	3/16	0.760	1.171	−0.411	0.83

75% $\Rightarrow \kappa \approx 7.6$; 50% $\Rightarrow \kappa \approx 3.0$; 19% $\Rightarrow \kappa \approx 0.83$. More accessible spectrum means weaker apparent correlation.

3. ***H*-function not required at tested level:** Simple κ-addition ($H = \mathrm{id}$) suffices to represent the tested toy cases at the fitted-parameter level. This does not establish uniqueness of the simple κ-form.

4. **Three-subsystem tests:** For tripartite systems (8-dimensional), direct bipartite splits (e.g., A vs BC) satisfy κ-addition with positive κ. However, marginal-of-marginal splits (e.g., A vs B within the AB marginal of a truncated ABC state) can yield negative κ values, corresponding to super-additive entropy. These cases arise from inherited truncation that does not correspond to a direct spectral threshold on the subsystem Hamiltonian.

Conjecture 2.4 (Sub-Additivity from Spectral Truncation). For any product thermal state $\rho = \rho_A \otimes \rho_B$ on a composite Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, and any spectral truncation Π_T defined by a threshold on the eigenvalues of the product Hamiltonian $D_A \otimes \mathbf{1} + \mathbf{1} \otimes D_B$, the truncated entropy composition is sub-additive:

$$S(\rho_\theta) \leq S(\rho_\theta^A) + S(\rho_\theta^B), \tag{10}$$

with equality if and only if no eigenvalues are truncated. Whether the specific functional form of the defect is the κ-addition product $S_A \cdot S_B/\kappa$ or a more general sub-additive function remains to be determined analytically.

Remark 2.5 (Why sub-additivity is plausible). For product Hamiltonians, the eigenstates of D_{AB} are tensor products $|i\rangle_A|j\rangle_B$. Spectral truncation selects a subset V of these product basis states. The truncated state remains diagonal in the product basis: $\rho_\theta = \sum_{(i,j)\in V}(p_i q_j/Z_\theta)\,|ij\rangle\langle ij|$. The non-product geometry of the subspace V is the sole source of the entropy defect. The subspace V is defined by the condition $|\lambda_i + \mu_j| \geq \theta$, which couples the A and B indices through the threshold, creating apparent correlation even for an initially uncorrelated state.

Remark 2.6 (Physical interpretation). The computation reveals the mechanism behind κ-addition:

1. **Full spectrum:** When all eigenvalues are accessible ($\theta = 0$), the thermal state of a product Hamiltonian is a product state. Entropy is additive. $\kappa \to \infty$.
2. **Truncated spectrum:** Spectral truncation removes eigenstates from the accessible sector. Even for an initially uncorrelated (product) state, the truncation creates apparent correlation by projecting onto a subspace that is not itself a product space.
3. **The κ parameter:** The strength of the apparent correlation is measured by κ. Stronger truncation (fewer kept eigenvalues) gives smaller κ (stronger non-additivity). This is precisely the v7/v8 interpretation: κ measures how much of the correlation structure is invisible to the observer at temperature T.
4. **The asymmetric cases are non-degenerate:** When subsystem couplings differ ($a \neq b$) and partial truncation removes some but not all mixed-sector eigenvalues, κ takes finite values that depend nontrivially on the thermal state and truncation level.

Independent κ Test (Cycle 2)

To test whether the κ-addition form has structural content beyond a one-parameter fit, we performed two additional analyses.

Cross-β stability test. At fixed truncation (keeping 4/9 eigenvalues of the 3-site chain composite), the effective κ was computed across seven values of β. An alternative model (defect $= -\alpha(S_A + S_B)$) was fitted in parallel.

Result: Neither κ nor α is constant across β. The coefficient of variation for α (0.36) is lower than for κ (0.55). This means the alternative sub-additive model fits at least as stably as κ-addition. The true functional form of the truncation-induced defect is more complex than either simple parametrisation.

Multi-system scan. We collected 28 data points across 2-site, 3-site, and 4-site chain composites (4 to 16 dimensions), varying β and truncation threshold. Key observations:

1. At the same truncation fraction (50%), κ ranges from 0.09 (2-site, $\beta = 1$) to 103.6 (4-site, $\beta = 2$). The κ parameter is **not** determined by the truncation fraction alone.

2. The direction of κ versus β **reverses** depending on truncation severity. For heavy truncation (keeping $\leq 25\%$), κ decreases with β. For light truncation (keeping $\geq 50\%$), κ increases with β. This is physically interpretable: it depends on the overlap between the truncation mask and the thermal occupation distribution.

3. No simple ratio $I(A{:}B)/f(S_A, S_B)$ is constant across β at fixed truncation. The mutual information created by truncation is a complex function of all parameters.

Remark 2.7 (What this means for the v7 bridge). The qualitative mechanism of the v7 bridge conjecture is confirmed: spectral truncation creates non-extensive composition, and the strength depends on the severity of the truncation. The specific κ-addition form (defect $\propto S_A \cdot S_B$) is consistent with the data but is not uniquely selected. Two possibilities remain:

1. The exact κ-addition law emerges in a regime not yet tested (e.g., the large-system limit, or for specific spectral geometries matching the expander graph structure).

2. The actual composition law from spectral truncation is a generalisation of κ-addition that reduces to it in specific limits.

Both possibilities are consistent with the data. Neither is confirmed. The quantitative bridge between the $\mathcal{W}$-atlas and thermodynamic relativity remains open but its qualitative mechanism is supported at the tested toy/scaling level.

Analytical Result: The Truncation Mutual Information

Proposition 2.8 (Exact Mutual Information from Spectral Truncation)**.** *Let $\rho = \rho_A \otimes \rho_B$ be a product thermal state with weights $p_i = e^{-\beta\lambda_i}/Z_A$ and $q_j = e^{-\beta\mu_j}/Z_B$ in the eigenbases of D_A and D_B respectively. Let $V = \{(i,j) : |\lambda_i + \mu_j| \geq \theta\}$ be the spectral truncation mask. Define:*

$$Z_V := \textstyle\sum_{(i,j)\in V} p_i\, q_j \qquad \textit{(retained thermal weight)}, \tag{11}$$

$$Q_i := \textstyle\sum_{j\in V(i)} q_j \qquad \textit{(B-weight accessible from A-state i)}, \tag{12}$$

$$P_j := \textstyle\sum_{i\in V(j)} p_i \qquad \textit{(A-weight accessible from B-state j)}. \tag{13}$$

Then the mutual information of the truncated state is exactly:

$$\boxed{I(A{:}B) = \ln Z_V - \frac{1}{Z_V}\sum_i p_i\, Q_i\, \ln Q_i - \frac{1}{Z_V}\sum_j q_j\, P_j\, \ln P_j.} \tag{14}$$

Proof. The truncated state $\rho_\theta = \sum_{(i,j)\in V}(p_i q_j/Z_V)\,|ij\rangle\langle ij|$ is diagonal in the product eigenbasis. Direct computation of $S(\rho_\theta)$, $S(\mathrm{Tr}_B\,\rho_\theta)$, and $S(\mathrm{Tr}_A\,\rho_\theta)$ yields (14). Verified numerically against matrix computation for 4×4 and 16×16 systems. □

Corollary 2.9 (Product Truncation Criterion)**.** *$I(A{:}B) = 0$ if and only if V is a product set ($V = V_A \times V_B$), in which case Q_i is constant across all $i \in V_A$ and P_j is constant across all $j \in V_B$.*

The mutual information measures the non-product geometry of the truncation mask: Q_i varies across i precisely when the mask couples the A and B indices through the threshold condition, creating apparent correlation in an initially uncorrelated state.

Remark 2.10 (Physical interpretation)*.* The quantities Q_i and P_j measure **spectral accessibility asymmetry**: how much of subsystem B is "visible" to an observer in eigenstate $|i\rangle_A$, and vice versa. When the truncation threshold couples the two subsystems (as spectral truncation generically does), different eigenstates of A see different amounts of B. This asymmetry is the source of the apparent correlation, and it is quantified exactly by (14).

Remark 2.11 (Epistemic status)*.* The computation is a **numerical result across multiple system sizes**, not a general analytical proof. It establishes that spectral truncation produces sub-additive entropy composition from 4-dimensional through 16-dimensional composite systems. Conjecture 2.4 proposes the analytical generalisation of the sub-additivity. The specific functional form of the defect (whether κ-addition or a more general sub-additive law) and the extension to non-product Hamiltonians remain open (O38).

Remark 2.12 (Connection to Livadiotis–McComas)*.* The result confirms the v7 bridge conjecture's **qualitative mechanism**: spectral coarse-graining creates non-extensive entropy composition, and the strength of non-extensivity scales with the severity of the coarse-graining. This is consistent with the Livadiotis–McComas identification of $1/\kappa$ with the strength of inter-particle correlations.

However, the computation does not uniquely select the specific κ-addition form. The **quantitative** bridge — the emergence of the exact H-function composition law from spectral truncation — remains open (O38).

The O1 Inverse Problem: Structure and a Critical Correction

The Problem

Open Problem O1 (v1) asks for the derivation of the off-diagonal metric component $g_{IE}(I, E)$ completing the spatial metric:

$$h_{ab} = \begin{pmatrix} g_{II} & g_{IE} \\ g_{IE} & g_{EE} \end{pmatrix} = \begin{pmatrix} \dfrac{\hbar^2}{4I} & g_{IE}(I,E) \\ g_{IE}(I,E) & \dfrac{4G\hbar}{k_\mathrm{B}^2 E^2} \end{pmatrix}. \tag{15}$$

A Critical Finding: The 2D EFE Constraint Is Vacuous

A natural attack on O1, proposed during the v13 planning cycle, was to solve "backward" from the requirement that the (I, E) Ricci tensor recovers the vacuum Einstein equations.

This approach fails for a fundamental reason.

In two dimensions, the Einstein tensor vanishes identically:

$$G_{\mu\nu} \equiv R_{\mu\nu} - \tfrac{1}{2} R\, g_{\mu\nu} \equiv 0 \qquad \text{(in 2D, for } \textit{any} \text{ metric).} \tag{16}$$

This is a standard result of differential geometry: in 2D, the Ricci tensor is fully determined by the scalar curvature, $R_{\mu\nu} = (R/2)\, g_{\mu\nu}$, making $G_{\mu\nu}$ identically zero.

Since the (I, E) sector is two-dimensional, the vacuum Einstein equations $G_{\mu\nu} = 0$ are trivially satisfied for *any* choice of $g_{IE}(I, E)$. The "backward EFE" constraint places no restriction on the unknown function.

The Actual Constraints on g_{IE}

The EFE recovery must operate on the full three-dimensional emergent spatial geometry, not the two-dimensional (I, E) base metric. The three spatial dimensions emerge through the replacement product construction of v3 (degree-3 graph from the cycle structure).

The constraints on $g_{IE}(I, E)$ are therefore:

1. **3D metric construction:** The function g_{IE} enters the three-dimensional spatial metric through the replacement product lift. The EFE recovery condition must be applied to the 3D metric, not the 2D base. This requires the explicit construction of the 3D metric from (h_{ab}, ϕ-sector).

2. **Determinant condition:** For three spatial dimensions to emerge (v1–v2), the de-

terminant of h_{ab} must approach zero in the classical limit $E \to E_{\max}$, $I \to I_{\max}$:

$$\det(h_{ab}) = \frac{G\hbar^3}{I\,k_B^2\,E^2} - g_{IE}^2 \to 0 \quad \Longrightarrow \quad |g_{IE}| \to \sqrt{g_{II}\,g_{EE}}. \tag{17}$$

This constrains the asymptotic behaviour of g_{IE} but not its full functional form.

3. **Ryu–Takayanagi compatibility:** The geodesic structure on the (I, E) surface, which depends on g_{IE}, must be compatible with the RT entanglement entropy formula (v1).

4. **Rotation map convergence:** The discrete rotation map Rot_{AP_q} of v3 must converge to g_{IE} in the continuum limit (O14, v3/v5).

5. **Spectral filter compatibility:** The function g_{IE} must be compatible with the monotone filtration (Proposition 3.1 of v12): the spatial metric induced at temperature T must be consistent with the accessible spectral sector U_T.

Remark 3.1 (Status of the O1 inverse programme)*.* The 2D vacuity finding eliminates the strongest proposed constraint from the inverse programme. The remaining constraints (items 1–5 above) are nontrivial but require the explicit 3D metric construction to be carried out. This construction depends on the replacement product geometry of v3, which is well-defined at the graph level but whose continuum limit has not been computed. O1 therefore remains open, but its constraint structure is now better understood.

Space–Time Complementarity

The Corollary

Corollary 4.1 (Space–Time Complementarity)**.** *Let $N_{\mathrm{acc}}(T)$ denote the number of eigenvalues of $D_{\mathcal{W}}$ in the accessible sector U_T (assuming discrete spectrum). Then:*

1. *$N_{\mathrm{acc}}(T)$ is monotonically non-increasing in T (from Proposition 3.1 of v12, Nested Accessibility).*

2. *The temporal processing rate $c_S(T) = k_B T/\hbar$ is monotonically increasing in T (from the v1 identification).*

Therefore, spatial spectral richness and temporal processing rate are inversely related across temperature:

$$T\uparrow \implies N_{\mathrm{acc}}\downarrow,\ c_S\uparrow \qquad \textit{and} \qquad T\downarrow \implies N_{\mathrm{acc}}\uparrow,\ c_S\downarrow. \tag{18}$$

Proof. Item (1) follows directly from v12 Proposition 3.1: if $T_1 < T_2$, then $U_{T_2} \subseteq U_{T_1}$, so $N_{\text{acc}}(T_2) \leq N_{\text{acc}}(T_1)$. Item (2) follows from the definition $c_S(T) = k_B T/\hbar$, which is manifestly increasing in T. The inverse relationship is the conjunction of these two monotone behaviours. □

Consequences at Cycle Endpoints

If the framework is applied to a cosmological setting, the complementarity corollary suggests asymptotic behaviour at the endpoints:

Hypothesis 4.2 (Cycle-Endpoint Asymptotics).

1. **Hot endpoint ($T \to T_P$):** $N_{\text{acc}} \to N_{\min}$ (minimal spatial spectral richness) while c_S reaches its maximum value $k_B T_P/\hbar = E_P/\hbar$ (maximal temporal rate). The observer reconstructs minimal spatial geometry at maximum temporal resolution.

2. **Cold endpoint ($T \to 0$):** $N_{\text{acc}} \to N_{\max}$ (full spectrum accessible) while $c_S \to 0$ (temporal coordinate degenerates). The observer has access to maximal spatial richness but can no longer generate temporal flow. This is the chart failure identified in v8.

Remark 4.3 (Epistemic status). The complementarity corollary (Corollary 4.1) is **Derived** from v12 propositions. The cycle-endpoint hypothesis (Hypothesis 4.2) is **Proposed**: it requires the framework to be applied to cosmological scales, which involves additional assumptions about the global structure of $D_{\mathcal{W}}$ not yet established.

The 3+1 Emergence Hypothesis

The O31 result, the O1 constraint analysis, and the complementarity corollary together support a structured 3+1 emergence hypothesis:

Hypothesis 5.1 (3+1 Emergence from Spectral Accessibility). Standard spacetime coordinates (x, y, z, t) are emergent reconstruction variables on the observer-accessible sector of the $\mathcal{W}$-atlas:

1. **Spatial coordinates** (x, y, z) arise from the observer's reconstruction of the accessible (I, E)-sector geometry, lifted to three dimensions through the replacement product cycle structure (v3). The effective spatial scale depends on the spectral sector accessible at the observer's temperature.

2. **The temporal coordinate** t arises as a chart-dependent ordering parameter associated with the entropy processing rate $c_S(T) = k_B T/\hbar$. Its effective resolution is set by the observer's thermal state.

3. **Observer dependence:** Different observer charts (different T) reconstruct different effective 3+1 descriptions from different accessible spectral sectors. In the matched-chart limit (v12, Theorem 3.6), these descriptions agree and no chart-mismatch distortion appears.

4. **Cosmological expansion** is reinterpreted: as the universe cools, the spectral filter Π_T widens (Proposition 3.1 of v12), the accessible (I, E) sector enriches, and the reconstructed spatial geometry gains resolution and scale.

Remark 5.2 (Epistemic status). This hypothesis is **Proposed**, not Derived. Items (1) and (4) depend on O1 being resolved. Item (2) is consistent with the v1 (S, ϕ) geodesic structure but has not been given a comparably rigorous emergence derivation. Item (3) follows from v12's Matched-Chart Consistency Theorem. A full derivation of the 3+1 map from the $\mathcal{W}$-atlas remains a frontier target.

Updated Open Problems

O1. Derivation of $g_{IE}(I, E)$. The naive "backward EFE" approach is vacuous in 2D (Section 2). The constraint must operate through the 3D replacement product lift. Status: *open, constraint structure clarified.*

O31. H-function composition from spectral truncation. Status: *qualitative mechanism supported at toy/scaling level; exact composition law remains open (Section 2).*

O37. **3D metric from replacement product lift.** Construct the explicit three-dimensional spatial metric from the (I, E) base metric h_{ab} and the v3 replacement product cycle structure. This is required before O1 can be constrained by EFE recovery.

O38. **O31 functional form determination.** Determine analytically whether the sub-additive defect from spectral truncation of product thermal states takes the specific κ-addition form ($\Delta S \propto S_A S_B$), or a different sub-additive form, or a more complex function of S_A, S_B, β, and θ. Test with non-product Hamiltonians and in the continuum limit.

O39. **Temporal emergence map.** Provide a rigorous construction of the temporal coordinate from the (S, ϕ) sector of the $\mathcal{W}$-atlas, comparable in precision to the spatial emergence programme for the (I, E) sector. Currently at the level of a working hypothesis (Hypothesis 5.1, item 2).

Table 2: The Manifold Relativity programme: series arc v1–v13.

Version	Date	Central Contribution
v1–v11	Mar–Apr 2026	(Under historical title "Entropy Waves".) $\mathcal{W}$-manifold, discrete substrate, W-atlas, candidate operator, chart matching
v12	Apr 2026	**Series renamed to Manifold Relativity.** Programme reframing; structural propositions; candidate observables; falsification criteria
v13	Apr 2026	Sub-additivity from spectral truncation (O31 qualitative validation); O1 constraint clarification (2D EFE vacuity); exact mutual information formula; Space–Time Complementarity Corollary; 3+1 emergence hypothesis

Series Arc

Conclusion

This edition advances the programme's frontier in two directions: one substantial positive result (with an important caveat) and one corrective finding.

The positive result is O31. Spectral truncation of product thermal states creates sub-additive entropy composition across all tested system dimensions (4 through 16). The effective κ parameter increases monotonically with the fraction of spectrum retained, and the full-spectrum limit recovers standard additivity. This validates the v7 bridge conjecture's qualitative mechanism: spectral truncation creates apparent correlation by projecting product states onto non-product subspaces, and the strength of the apparent correlation scales with the severity of the truncation.

The caveat: The specific κ-addition functional form (defect $\propto S_A \cdot S_B$) is not uniquely selected by the computation. An alternative sub-additive model (defect $\propto S_A + S_B$) fits the data comparably well. Neither parameter (κ nor α) is constant across β at fixed truncation. The exact functional form of the truncation-induced defect remains open. O31 is partially addressed — the mechanism works qualitatively — but the quantitative bridge to thermodynamic relativity's specific composition law is not yet closed.

The corrective finding is on O1. The "backward EFE" approach to deriving g_{IE} is vacuous in two dimensions because the Einstein tensor vanishes identically for any 2D metric. The actual constraints on g_{IE} must operate through the three-dimensional replacement product lift (new Open Problem O37). This redirects the O1 programme from a naive inverse PDE to a more structurally informed construction problem.

The Space–Time Complementarity Corollary is a structural observation derived from v12's propositions. It is not deep mathematics, but it has physical content: it formalises the inverse relationship between spatial richness and temporal rate across temperature, with specific consequences for cycle-endpoint behaviour.

The 3+1 emergence hypothesis organises the programme's spatial and temporal claims into a unified statement. It is Proposed, not Derived. Its derivation is the long-term target of the programme.

The frontier after v13:

- O37 (3D metric construction) enables a non-vacuous O1 attack.
- O38 (O31 scaling) determines whether the κ-addition mechanism generalises.
- O39 (temporal emergence) closes the 3+1 programme on the time side.

We preserve all computational outcomes, including misleading first-pass successes, weaker alternative fits, and negative uniqueness checks, as supplementary addenda rather than only the final positive claims. The computational evidence trail (five Python scripts with raw outputs) is available as a downloadable addendum alongside this preprint.

References

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